Algebraic curves
Solutions sheet 7

May 1, 2024

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let X be a topological space and $Y \subseteq X$ a topological subspace. Show the following assertions:

1. $dim(Y) \leq dim(X)$.
2. If $(U_i)_{1\leq i\leq n}$ is an open cover of X, then $dim(X) = sup_{1\leq i\leq n}\,(dim(U_i))$.
3. Find an example of $Y \subseteq X$ such that Y is open and dense in X and $dim(Y) < dim(X)$. (Hint: Consider a topological space consisting of two points, with only one of them being closed).
4. If X is irreducible, has finite dimension and Y is closed, then $dim(Y) = dim(X) \Leftrightarrow Y = X$.

Solution 1.

1. A length n chain of closed irreducible subsets in Y gives rise to a length n chain of irreducible subsets in X. It suffices to take the closure in X of each set in the chain. We need to check that the inclusion remains strict after taking closure in X. It follows from the observation that if Z is irreducible and closed in Y, and $\bar{Z}$ its closure in X, then $\bar{Z} \setminus Z \subset X \setminus Y$ because $Z = \bar{Z} \cap Y$.
2. From 1), for each U_i, $dim(U_i) \leq dim(X)$. For the reverse inequality, take a chain of length n in X

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

   There exists i such that $Z_0 \cap U_i \neq \emptyset$. Then, using that non-empty open sets are dense in irreducible sets and two of them must intersect, we get for all j

$$Z_{j+1} \setminus Z_j \cap U_i \neq \emptyset$$

   Hence we get a chain with strict inclusions of closed irreducible sets in U_i.

$$Z_0 \cap U_i \subset Z_1 \cap U_i \subset \cdots \subset Z_n \cap U_i$$

   Thus, chain of length n in X gives rise to a chain of length n in one of the U_i, which allows to conclude that $dim(X) \leq \underset{i}{sup}(dim(U_i))$.

3. As suggested, take $X = \{x, y\}$, with the topology $\mathcal{T} = \{X, \emptyset, \{x\}\}$. $Y = \{x\}$ is open and dense in X, of dimension 1. X has dimension 1 since

$$\{x_1\} \subset X$$

is a chain of length 1. This is actually an important example in algebraic geometry, since it happens with local rings. Indeed, *Spec* $\mathcal{O}_F$ with Zariski topology has two points, given by the ideals (0) and $\mathfrak{m}$. (0) is open and dense.

4. Clearly if $Y = X$ it is always true that $dim(Y) = dim(X)$. Now, assume X irreducible, finite dimensional and Y closed wtih $dim(Y) = dim(X) = n$. Suppose $X \setminus Y \neq \emptyset$. Take a chain of length n in Y, $Y_0 \subset \cdots \subset Y_n$. Since X is irreducible, we get a length $n+1$ chain in X :

$$Y_0 \subset \cdots \subset Y_n \subset X$$

This is a contradiction with $dim(X) = n$. Hence $X = Y$.

Exercise 2. Let V, W be algebraic varieties and $P \in V$. Show the following assertions:

1. $dim(V) = 0 \Leftrightarrow$ V is finite. (Hint: reduce to the affine case via an open affine cover of V).

2. If V and W are affine, then $dim(V \times W) = dim(V) + dim(W)$. (Hint: by Noether normalization, the transcendence degree of $k(V)$ is also the cardinality of a maximal subset of algebraically independent elements of $\Gamma(V)$).

Solution 2.

1. Let V be an algebraic variety of dimension 0. We can reduce to affine case, since there is always a finite quasi-affine $V = \cup_i U_i$ cover of V, and thanks to exercice 1, question 2, we know that each quasi-affine open U_i of V is also of dimension 0. A closure $\overline{U}_i$ is also necessary of dimension 0. If each of those $\overline{U}_i$ is a finite set of points, V is a finite union of finite set of points, hence a finite set of points.

   Suppose V affine, $V = \mathsf{Spec}\ k[x_1, \ldots, x_n]/I$, I reduced. Using results of the course we get that $ht(I) = dim(k[x_1, \ldots, x_n])$, hence I is maximal, hence V is a point.

2. $\Gamma(V)$ is an integral domain, hence the Krull dimension is equal to the transcendance degree of $k(V)$. Choose a transcendance basis for $k(V)$ contained in $\Gamma(V)$,

$$\Gamma(V) = k[T_1, \ldots, T_n]/I$$

   Same for W,

$$\Gamma(W) = k[S_1, \ldots, S_m]/J$$

   Now putting together maximal algebraically independant subsets of $\Gamma(V)$ and of $\Gamma(W)$ gives a maximal algebraically independant subset of $\Gamma(V \times W)$.

$$\Gamma(V \times W) = k[T_1, \ldots, T_n, S_1, \ldots, S_m]/I \cdot \Gamma(W) + J \cdot \Gamma(V)$$

Exercise 3. Let $Y = \{(t^3, t^4, t^5),\ t \in k\} \subseteq \mathbb{A}^3_k$. We denote by $ht(I)$ the height of an ideal I.

1. Show that Y is a subvariety of $\mathbb{A}^3_k$ and compute $I(Y)$.

2. Compute $r = ht(I(Y))$. Show that $I(Y)$ cannot be generated by r elements. (Hint: Show that $dim_k(I(Y)/(x,y,z)I(Y)) \geq 3$).

Solution 3.

1. The morphism

$$\begin{aligned} f \ : \ \mathbb{A}^1_k &\longrightarrow \mathbb{A}^3_k \\ t &\longmapsto (t^3, t^4, t^5) \end{aligned}$$

has Y as image. It gives a description of the ring of functions of Y

$$\begin{aligned} f^* \ : \ k[x,y,z] &\to k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^4 \\ z &\longmapsto t^5 \end{aligned}$$

Hence,

$$\mathcal{O}(Y) = k[x,y,z]/\ker f^* \simeq k[t^3,t^4,t^5]$$

Note that $k[t^3,t^4,t^5] = \{f \in k[x], f'(0) = f''(0)\}$. As $\mathcal{O}(Y)$ is integral, we see that $I(Y)$ is prime, so Y is a subvariety. We compute $I(Y) = \{y^2 - xz, x^2 - yz, z^2 - x^2y\}$.

2. As $k(Y) = k(t)$, $dim Y = 1$ hence $ht(I(Y)) = codim(Y) = 2$. Suppose $I(Y) = (f,g)$. This is a surjection

$$k[x,y,z]^{\oplus 2} \to I(Y), (a,b) \mapsto a \cdot f + b \cdot g$$

It implies a surjection

$$k^2 = (k[x,y,z]/(x,y,z))^{\oplus 2} \to I(Y)/(x,y,z)I(Y)$$

However, using f^*, we see that $I(Y)$ maps to polynomials in t of degree $8, 9, 10$ while $(x,y,z)I(Y)$ maps to polynomials in t of degree at least 11. Hence the $k-$vector space $I(Y)/(x,y,z)I(Y)$ has dimension at least 3, and this contradicts the surjection.

Exercise 4. We denote by $k(V)$ the field of fractions of an algebraic variety V. Let $n \geq 1$.

1. Compute $k(\mathbb{P}^n_k)$.

2. Let $V_1 = V(y^2 - x^3) \subseteq \mathbb{A}^2_k$. Show that $k(V_1) \simeq k(\mathbb{P}^1_k)$. Is V_1 isomorphic to $\mathbb{P}^1_k$?

3. Let $V_2 = V_P(x_1x_2 - x_3x_4) \subseteq (\mathbb{P}^3_k$. Show that $k(V_2) \simeq k(\mathbb{P}^2_k)$. Is V_2 isomorphic to $\mathbb{P}^2_k$? (Hint: You may assume that any two curves in $\mathbb{P}^2_k$ intersect).

Solution 4.

1. $k(\mathbb{P}^n_k) = k(U_0) = k(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$

2. $k(V_1) = Frac(k[x^2,x^3]) = k(x) = k(\mathbb{P}^1_k)$. However they are not isomorphic because one is affine and not the other.

3. $k(V_2) = k(V_2 \cap U_1)$. $V_2 \cap U_1 = V(x_2 - x_3x_4)$. $k[V_2 \cap U_1] = k[x_3, x_4]$ so that $k(V_2) = k(\mathbb{P}^2_k)$. We observe that $V_P(x_1, x_3) \cap V_P(x_2, x_4) = \emptyset$, and those algebraic sets are curves in V_2.

Exercise 5. Let $X = \mathbb{P}^2_k \setminus \{x\}$.

1. $\mathcal{O}(X)$? $k(X)$?

2. X is not quasi affine nor projective.

Solution 5.

1. $\mathcal{O}(X) = k$, since any non constant function would have a one dimensional pole set. If $x \in U_i$, $k(X) = k(U_i \cap X) \simeq k(\mathbb{A}^2 \setminus \{0, 0\}) = k(x, y)$, as seen in a previous exercise.

2. It is not quasi-affine as $\mathcal{O}(X) = k$ (and it is clearly not just one point). If X is projective, let $P \xrightarrow{i} X$ be the isomorphism, with $P \hookrightarrow \mathbb{P}^n$ closed. Then we get

$$P \xrightarrow{i} X \hookrightarrow \mathbb{P}^2$$

hence X is closed, which is a contradiction.